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Optimal Approximation: an Addendum

ARTHUR SARD

Queens College, The City University of New York, Flushing, New York

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In my paper [3], I consider the approximation of Gx, $x \in X$, in terms of observations $F^i x$, $i = 1, \ldots, m$, where X is a Banach space and G, F^i are given linear continuous operators on X to a Banach space Z and Hilbert spaces Z^i, respectively. The observations are sufficient to determine the approximation Ax of Gx but are not sufficient to determine the error $Rx = Gx - Ax$. One assumes the existence of a linear continuous operator U on X onto an inner product space Y such that, for some constant $B < \infty$,

$$\| x \|^2 \leqslant B^2 \left[\| Ux \|^2 + \sum_{i=1}^{m} \| F^i x \|^2 \right] \qquad \text{for all} \quad x \in X.$$

One assumes also that the ranges $F^i X$ are closed. It follows that the observations and Ux determine Rx and that a particular approximation A_0 exists, called the optimal approximation of G relative to U and the F^i, with attractive minimal and interpolatory properties. Details are in [3]. The notation and definitions of [3] are in force throughout the present note.

There is an additional property of A_0, due to Golomb and Weinberger [1] and extended to related problems by Meinguet [2], which is described in the following paragraph.

Let Γ be the subset of X compatible with the observations and the condition $\| Ux \| \leqslant d$. That is, data $a^i \in Z^i$, $i = 1, \ldots, m$, and $d \geqslant 0$ are given, and

$$\Gamma = \{x \in X : F^i x = a^i, \ i = 1, \ldots, m, \text{ and } \| Ux \| \leqslant d\}.$$

Assume that Γ is not empty. Then $G\Gamma = \{Gx : x \in \Gamma\}$ is a bounded convex subset of Z with center; the center of $G\Gamma$ is dependent on $a^1, \ldots, a^m$ but independent of d; the center of $G\Gamma$ is $G\xi$, where $\xi = \mathrm{Proj}_M x$ is the spline approximation of $x \in \Gamma$, ξ being the same for all elements x of Γ. And

$$\| Gx - G\xi \|^2 \leqslant \| G \restriction N \|^2 [d^2 - \| U\xi \|^2], \qquad x \in \Gamma, \tag{1}$$

with equality approachable. Since $G\xi$ is the center of $G\Gamma$, $G\xi$ is the best approximation of Gx, $x \in \Gamma$, in the following sense: If $\beta \in Z$ and $\beta \neq G\xi$, then $\| Gx - \beta \|^2$ attains values greater than the bound in (1) for some $x \in \Gamma$. Thus $A_0 x = G\xi = G \operatorname{Proj}_M x$ is the best approximation of Gx, $x \in \Gamma$, and therefore of Gx, $x \in X$. And

$$\| Gx - A_0 x \|^2 \leqslant \| G \upharpoonright N \|^2 [\| Ux \|^2 - \| U\xi \|^2], \qquad \xi = \operatorname*{Proj}_M x,$$

for all $x \in X$.

The proof of these assertions is like that of Golomb and Weinberger for the scalar case [*1*], [*2*].

The optimal approximation A_0 and its appraisals are the same as before. Whether one's discussion starts with the present aspect of A_0 or the definition of [*3*], one would show that $A_0 \in \mathscr{A}$ as in [*3*] and one could end with the same ensemble of properties.

References

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2. Meinguet, J., Optimal approximation and error bounds in seminormed spaces. *Numer. Math.* **10** (1967), 370–388.

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